Approximate Osher-Solomon schemes for hyperbolic systems

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Outline

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2. PVM and RVM methods
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   - RVM methods
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Preliminaries

- Consider a hyperbolic system of conservation laws

$$\boxed{\partial_t U + \partial_x F(U) = 0}$$

where $U(x,t)$ takes values on an open convex set $\mathcal{O} \subset \mathbb{R}^N$ and $F\colon \mathcal{O} \to \mathbb{R}^N$ is a smooth flux function.

- We are interested in the numerical solution of the Cauchy problem for the system by means of a finite volume method of the form

$$\boxed{U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)}$$

where U_i^n denotes the approximation to the average of the exact solution at the cell $I_i = [x_{i-1/2}, x_{i+1/2}]$ at time $t^n = n\Delta t$.

- We consider numerical fluxes that are defined as (we drop the dependence on time)

$$\boxed{F_{i+/2} = \frac{F(U_i) + F(U_{i+1})}{2} - \frac{1}{2} Q_{i+1/2}(U_{i+1} - U_i)}$$

where $Q_{i+1/2}$ is a **numerical viscosity matrix**. Different numerical methods can be designed depending on the choice of the viscosity matrix.

Preliminaries

Examples:

- Roe:

$$Q_{i+1/2} = |A_{i+1/2}|$$

where $A_{i+1/2}$ is a Roe matrix for the system.

- Lax-Friedrichs:

$$Q_{i+1/2} = \frac{\Delta x}{\Delta t} I$$

being I the identity matrix.

- Lax-Wendroff:

$$Q_{i+1/2} = \frac{\Delta t}{\Delta x} A^2_{i+1/2}$$

- FORCE and GFORCE:

$$Q_{i+1/2} = (1-\omega)\frac{\Delta x}{\Delta t} Id + \omega \frac{\Delta t}{\Delta x} A^2_{i+1/2}$$

with $\omega = \frac{1}{2}$ and $\omega = \frac{1}{1+\text{CFL}}$, respectively.

Preliminaries

We propose a class of finite volume methods defined by

$$\boxed{Q_{i+1/2} = f(A_{i+1/2})}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a function and $A_{i+1/2}$ is a Roe matrix or the Jacobian of the flux evaluated at some average state.

Some properties of f:

- $f(x) \geqslant 0$ and smooth.
- $f(A_{i+1/2})$ should be **easy** to evaluate: no spectral decomposition of $A_{i+1/2}$ needed.
- L^∞ linear stability:

  $$\text{CFL}\,\frac{\Delta x}{\Delta t} \geqslant f(x) \geqslant |x|, \quad \forall\, x \in [\lambda_1^{i+1/2}, \lambda_N^{i+1/2}],$$

  where $\lambda_l^{i+1/2}$ are the eigenvalues of $A_{i+1/2}$.
- $f(x)$ should be as close as possible to $|x|$.

PVM methods

PVM methods

One possible choice is to set

$$f(x) = P_d(x),$$

being $P_d(x)$ a polynomial of degree d.

In this case

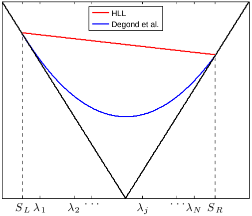
$$\boxed{Q_{i+1/2} = P_d(A_{i+1/2})}$$

that is, $Q_{i+1/2}$ is a Polynomial Viscosity Matrix (PVM).

[Castro-Fernández Nieto, SIAM J. Sci. Comput. 34, 2012]

- PVM methods has been applied to multilayer shallow water equations or the two-phase flow model of Pitman-Le, for which the eigenstructures **are not explicitely known**.
- PVM methods can be extended to **nonconservative hyperbolic systems**, following the theory of path-conservative schemes ([Pares, SIAM J. Numer. Anal. 44, 2006]).
- Some well-known solvers as Lax-Friedrichs, Rusanov, FORCE/GFORCE, HLL, Roe, Lax-Wendroff, etc., can be recovered as PVM methods. In particular, this allows to build direct extensions of the mentioned solvers to the nonconservative case.

PVM methods



Some examples:

- Lax-Friedrichs, modified Lax-Friedrichs and Rusanov (local LxF):

$$P_0(x) = S_0, \qquad S_0 \in \{S_{\mathrm{LF}}, S_{\mathrm{LF}}^{\mathrm{mod}}, S_{\mathrm{Rus}}\}$$

  with $S_{\mathrm{LF}} = \frac{\Delta x}{\Delta t}$, $S_{\mathrm{LF}}^{\mathrm{mod}} = CFL\frac{\Delta x}{\Delta t}$ and $S_{\mathrm{Rus}} = \max_j |\lambda_j^{i+1/2}|$.

- HLL:

$$P_1(x) = \alpha_0 + \alpha_1 x$$

  such that $P_1(S_L) = |S_L|$, $P_1(S_R) = |S_R|$, S_L and S_R being approximations to the minimum and maximum wave speeds.

- FORCE:

$$P_2(x) = \alpha_0 + \alpha_2 x^2$$

  such that $P_2(S_0) = S_0$ and $P_2'(S_0) = 1$, with $S_0 = S_{\mathrm{LF}}$.

- The related solver proposed in [Degond et al., C.R. Acad. Sci. Paris Sér. I 328, 1999] can be viewed as a PVM method based on a second order polynomial.

- The incomplete Riemann solver based on Krylov subspace approximations of $|x|$ in [Torrilhon, SIAM J. Sci. Comput. 34, 2012] can also be interpreted as a PVM scheme.

PVM methods

PVM-Force type iterative method

Consider the polynomials defined as follows

- $$P_0(x) = 1, \quad P_n(x) = P_{n-1}(x) - \frac{P_{n-1}^2(x) - x^2}{2}, \quad n = 1, 2, \dots$$
- Viscosity matrix:

$$\boxed{Q_{i+1/2} = |\lambda_{\max}| P_n\left(\frac{1}{|\lambda_{\max}|} A_{i+1/2}\right) \approx |A_{i+1/2}|}$$

  where $\lambda_{\max}$ is the eigenvalue of $A_{i+1/2}$ with maximum modulus.
- Observe that the viscosity matrix obtained with $n = 1$ is

$$Q_{i+1/2} = \frac{\lambda_{\max}}{2} Id + \frac{1}{2\lambda_{\max}} A_{i+1/2}^2.$$

PVM methods

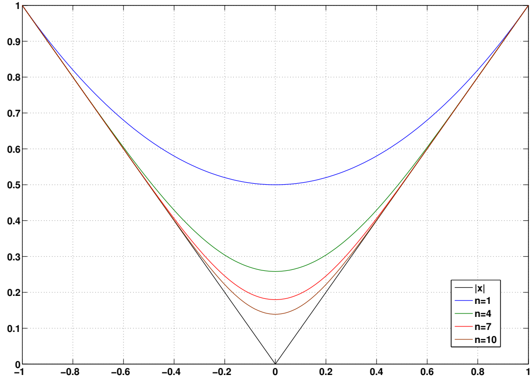


Figure: PVM-Force type iterative method: $P_n(x)$ for $n = 1,\ 4,\ 7$ and 10.

PVM methods

PVM-Chebyshev

- Chebyshev polynomials provide optimal uniform approximation to $|x|$ in $[-1,1]$:

$$|x| = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{4}{\pi} \frac{(-1)^{k+1}}{(2k-1)(2k+1)} T_{2k}(x), \quad x \in [-1,1],$$

  where the Chebyshev polynomials of even degree $T_{2k}(x)$ are recursively defined as

$$T_0(x) = 1, \quad T_2(x) = 2x^2 - 1, \quad T_{2k}(x) = 2T_2(x)T_{2k-2}(x) - T_{2k-4}(x).$$

- Viscosity matrix:

$$\boxed{Q_{i+1/2} = P_{2p}(A_{i+1/2}) = |\lambda_{\max}|\tau_{2p}\left(\frac{1}{|\lambda_{\max}|}A_{i+1/2}\right) \approx |A_{i+1/2}|}$$

  where $\lambda_{\max}$ is the eigenvalue of $A_{i+1/2}$ with maximum modulus.

[Castro, Gallardo, Marquina, J. Sci. Comput. 60, 2014]

PVM methods

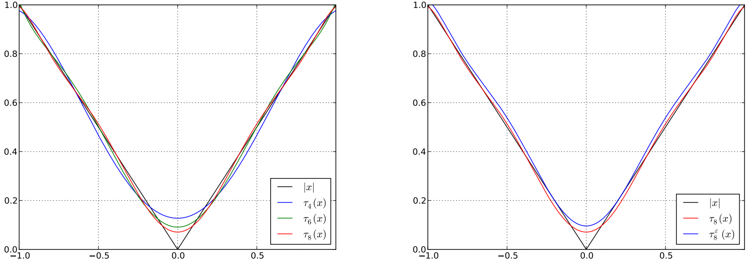


Figure: Left: Chebyshev approximations $\tau_{2p}(x)$ for $p = 2, 3, 4$. Right: $\tau_8(x)$ and $\tau_8^\varepsilon(x)$.

- Notice that $\tau_{2p}(x)$ do not satisfy the stability condition $\tau_{2p}(x) \geqslant |x|$. This drawback can be avoided by using $\tau_{2p}^\varepsilon(x) = \tau_{2p}(x) + \varepsilon$ such that $\tau_{2p}^\varepsilon(x) \geqslant |x|$.

PVM methods

PVM-Sign approximations

We could also consider methods based on the polynomial approximation of the sign function to define an approximation of $|A_{i+1/2}|$.

- Let us consider the Newton-Schulz iterative procedure to approximate the sign of x, $x \in [-1, 1]$

  $$x_0 = x, \quad x_n = \frac{x_{n-1}}{2}\left(3 - x_{n-1}^2\right), n = 1, 2, 3, \ldots$$

  then we could define

  $$\boxed{Q_{i+1/2} = A_{i+1/2} P_n \left(\frac{1}{|\lambda_{\max}|} A_{i+1/2}\right) \approx |A_{i+1/2}|}$$

  where $P_n(x)$ is the polynomial defined by

  $$P_0(x) = x, \quad P_n(x) = \frac{P_{n-1}(x)}{2}\left(3 - P_{n-1}^2(x)\right), \ n = 1, 2, \ldots$$

PVM methods

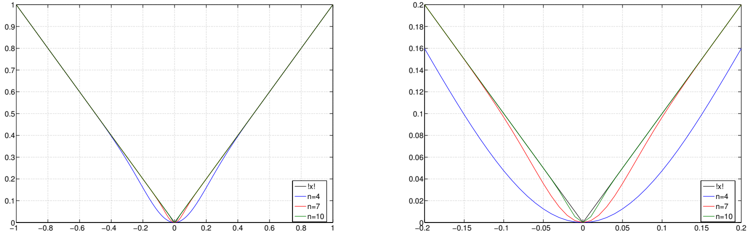


Figure: Left: PVM sign approximation for $n = 4, 7$ and 10. Right: PVM sign approximation Zoom at $[-0.2, 0.2]$

PVM-Chebyshev Jacobian free method

Chebyshev Jacobian free implementation

- Note that the viscosity matrix $Q_{i+1/2}$ need not to be computed explicitly, but only the vector $Q_{i+1/2}\Delta U$, where $\Delta U = U_{i+1} - U_i$:

$$\boxed{Q_{i+1/2}\Delta U = |\lambda_{\max}|\left(\alpha_0 \Delta U + \sum_{j=1}^{p} \alpha_j U^{[2j]}\right)}$$

with $\alpha_0 = \frac{2}{\pi}$, $\alpha_j = \frac{4}{\pi}\frac{(-1)^{j+1}}{(2j-1)(2j+1)}$ for $j \geqslant 1$ and

$$U^{[2j]} = T_{2j}\left(|\lambda_{\max}|^{-1} A_{i+1/2}\right)\Delta U.$$

PVM-Chebyshev Jacobian free method

Chebyshev Jacobian free implementation

- From the definition of Chebyshev polynomials, $U^{[2j]}$ can be recursively defined as
  - $U^{[0]} = \Delta U.$
  - $U^{[2]} = 2|\lambda_{\max}|^{-2} A_{i+1/2}^2 \Delta U - \Delta U.$
  - $U^{[2j]} = 4|\lambda_{\max}|^{-2} A_{i+1/2}^2 U^{[2j-2]} - 2U^{[2j-2]} - U^{[2j-4]}, \quad \text{for } j \geqslant 2.$
- The above expressions allow an efficient implementation of the PVM-Chebyshev method.
- In some cases the Jacobian $A_{i+1/2}$ may be difficult or expensive to compute. It is then interesting to implement the recursive form of the scheme without explicitly computing $A_{i+1/2}$.

PVM-Chebyshev Jacobian free method

Chebyshev Jacobian free implementation

- The finite difference formulation

$$A(U)V = \frac{\partial F}{\partial U}(U)V = \lim_{\varepsilon \to 0} \frac{F(U+\varepsilon V) - F(U)}{\varepsilon} \approx \frac{F(U+\varepsilon V) - F(U)}{\varepsilon}$$

leads to the following approximation:

$$A(U)^2 V \approx \frac{F\big(U + F(U+\varepsilon V) - F(U)\big) - F(U)}{\varepsilon}$$

where the parameter ε must be small with respecto to the norm of U.

- Then, the vector $U^{[2j]}$ can be redefined as

$$\boxed{\begin{aligned} U^{[2j]} = & \frac{4}{\varepsilon|\lambda_{\text{max}}|^2}\Big(F\big(U_{i+1/2} + F(U_{i+1/2} + \varepsilon U^{[2j-2]}) - F(U_{i+1/2})\big) - F(U_{i+1/2})\Big) \\ & - 2U^{[2j-2]} - U^{[2j-4]} \end{aligned}}$$

where $U_{i+1/2}$ is an intermediate state between U_i and U_{i+1}.

RVM methods

RVM methods

- The order of approximation to $|x|$ can be greatly improved by using **rational functions** instead of polynomials.
- RVM (Rational Viscosity Matrix) methods are defined in the same way as PVM methods, but using rational functions instead of polynomials.
- Two families will be considered here:
  - ***Newman*** rational functions ([Newman, Michigan Math. J. 11, 1964]), defined as

$$R_r(x) = x\,\frac{p(x)-p(-x)}{p(x)+p(-x)}, \qquad p(x) = \prod_{k=1}^{r}(x+\xi_k)$$

    for a given set of nodes $0 < \xi_1 < \cdots < \xi_r \leqslant 1$.

  - ***Halley*** rational functions, recursively defined as

$$H_{r+1}(x) = H_r(x)\,\frac{H_r(x)^2+3x^2}{3H_r(x)^2+x^2}, \qquad H_0(x) = 1.$$

[Castro, Gallardo, Marquina, J. Sci. Comput. 60, 2014]

RVM methods

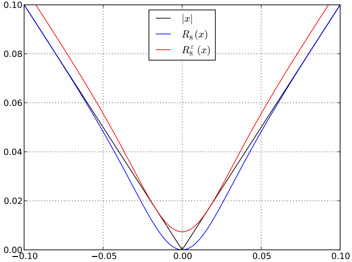


Figure: Left: Newman approximations $R_8(x)$ and $R_8^{\varepsilon}(x)$. Right: Halley functions $H_r(x)$, $r = 3, 4, 5$. Zoom at $[-0.1, 0.1]$

Approximate OS solvers

The Osher-Solomon (OS) method

- The ***Osher-Solomon scheme*** is a nonlinear and complete Riemann solver which enjoys a number of attractive features: it is robust, entropy-satisfying, smooth and well-behaved when computing slowly-moving shocks.

- **OS numerical flux**:

$$\boxed{F_{i+1/2} = \frac{F(U_i) + F(U_{i+1})}{2} - \frac{1}{2}\int_0^1 \left|A(\Phi(s))\right|\Phi'(s)ds}$$

  where $A = \frac{\partial F}{\partial U}$ is the Jacobian of F and Φ is a path linking U_i and U_{i+1} in phase space.

- It requires the computation of a path-dependent integral in phase space, which makes it very complex and computationally expensive. Due to this difficulties, its practical application has been restricted to certain systems, e.g., the compressible Euler equations.

Approximate OS solvers

The Dumbser-Osher-Toro (DOT) solver

- Dumbser and Toro ([Commun. Comput. Phys. 10, 2011]) have proposed a way to circumvent the drawbacks of the Osher-Solomon solver, maintaining at the same time its good features.

- The idea is to take Φ as the segment joining U_i and U_{i+1}, and then use a Gauss-Legendre quadrature formula to approximate the resulting integral.

- **DOT numerical flux**:

$$\boxed{F_{i+1/2} = \frac{F(U_i) + F(U_{i+1})}{2} - \frac{1}{2}\left(\sum_{k=1}^{q} \omega_k \left|A(U_i + s_k(U_{i+1} - U_i))\right|\right)(U_{i+1} - U_i).}$$

Approximate OS solvers

- The DOT numerical scheme is simple to implement and applicable to general hyperbolic systems, even in the nonconservative case.

- As a drawback, it requires the knowledge of the full eigenstructure of the system, in order to compute the intermediate matrices

$$|A(U_i + s_k(U_{i+1} - U_i))|, \quad k = 1, \dots, q.$$

- For systems in which the eigenstructure is not known or difficult to compute, the DOT scheme may be computationally expensive.

- We propose a new version of the DOT scheme in which the intermediate matrices are approximated in a simple and efficient way.
  [Castro, Gallardo, Marquina, Appl. Math. Comput. 272, 2016]

Approximate OS solvers

Approximate OS solvers

- The idea of the approximate OS solvers consists in using functional approximations to the absolute value function to compute the intermediate matrices, in the same spirit as the PVM and RVM methods previously introduced.

- Thus, if $P(x)$ ia a polynomial approximation to $|x|$ satisfying the stability condition, the ***polynomial approximate OS flux*** associated to $P(x)$ is defined as

$$\boxed{F_{i+1/2} = \frac{F(U_i) + F(U_{i+1})}{2} - \frac{1}{2}\left(\sum_{k=1}^{q} \omega_k \widetilde{P}^{(k)}_{i+1/2}\right)(U_{i+1} - U_i)}$$

where

$$\widetilde{P}^{(k)}_{i+1/2} = \left|\lambda^{(k)}_{i+1/2,\max}\right| P\left(\left|\lambda^{(k)}_{i+1/2,\max}\right|^{-1} A^{(k)}_{i+1/2}\right)$$

with

$$A^{(k)}_{i+1/2} = A(U_i + s_k(U_{i+1} - U_i)), \quad k = 1, \ldots, q.$$

- Obviously, ***rational approximate OS*** fluxes can be defined in a similar way.

Approximate Os solvers: Some examples

- Ideal magnetohydrodynamics (MHD) equations.
- Relativistic hydrodynamics (RHD) equations.
- Two-layer Savage-Hutter shallow-water model.

Approximate OS solvers: Some examples

- Ideal magnetohydrodynamics (MHD) equations:

$$\boxed{\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) & = 0 \\ \partial_t (\rho \mathbf{v}) + \nabla \cdot \left(\rho \mathbf{v}\mathbf{v}^T + \left(P + \frac{1}{2}\mathbf{B}^2 \right) I - \mathbf{B}\mathbf{B}^T \right) & = 0 \\ \partial_t \mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B}) & = 0 \\ \partial_t E + \nabla \cdot \left(\left(\frac{\gamma}{\gamma - 1} P + \frac{1}{2}\rho q^2 \right) \mathbf{v} - (\mathbf{v} \times \mathbf{B}) \times \mathbf{B} \right) & = 0 \end{cases}}$$

- Divergence-free condition: $\boxed{\nabla \cdot \mathbf{B} = 0}$
- Notations:
  - ρ: mass density; $\mathbf{v}$: velocity field; $\mathbf{B}$: magnetic field.
  - $E = \frac{1}{2}\rho q^2 + \frac{1}{2}B^2 + \rho\varepsilon$ is the total energy, where $q = \|\mathbf{v}\|$, $B = \|\mathbf{B}\|$ and ε denotes the specific internal energy.
  - Equation of state: $P = (\gamma - 1)\rho\varepsilon$; P: hydrostatic pressure; γ: adiabatic constant.

Approximate OS solvers: Some examples

The $\nabla \cdot \mathbf{B} = 0$ constraint

The divergence-free condition has been imposed using the the **nonconservative form** of the ideal MHD equations as in [Fuchs, Mishra, Risebro, J. Comput. Phys. 228, 2009]

$$\boxed{\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) & = 0 \\ \partial_t (\rho \mathbf{v}) + \nabla \cdot \left(\rho \mathbf{v} \mathbf{v}^T + \left(P + \frac{1}{2} \mathbf{B}^2 \right) I - \mathbf{B} \mathbf{B}^T \right) & = -\mathbf{B}(\nabla \cdot \mathbf{B}) \\ \partial_t \mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B}) & = -\mathbf{v}(\nabla \cdot \mathbf{B}) \\ \partial_t E + \nabla \cdot \left(\left(\frac{\gamma}{\gamma - 1} P + \frac{1}{2} \rho q^2 \right) \mathbf{v} - (\mathbf{v} \times \mathbf{B}) \times \mathbf{B} \right) & = -(\mathbf{v} \cdot \mathbf{B})(\nabla \cdot \mathbf{B}) \end{cases}}$$

The path-conservative framework is applied to this set of equations, where the r.h.s. can be interpreted as a source term.

Approximate OS solvers: Some examples

Brio-Wu shock tube problem

- Riemann problem for the 1d MHD system with initial data (Brio-Wu, 1988):

$$(\rho, v_x, v_y, v_z, B_x, B_y, B_z, P) = \begin{cases} (1, 0, 0, 0, 0.75, 1, 0, 1) & \text{for } x \leqslant 0, \\ (0.125, 0, 0, 0, 0.75, -1, 0, 0.1) & \text{for } x > 0, \end{cases}$$

  with $\gamma = 2$.

- $\Delta x = 1/500$, CFL=0.8, $T = 0.2$.
- **Schemes:** HLL, Roe, DOT, OS-Cheb-4, OS-Newman-4, and OS-Halley-2.
- **High-order methods**: third-order PHM in space and third-order TVD Runge-Kutta in time.

Approximate OS solvers: Some examples

Brio-Wu shock tube problem

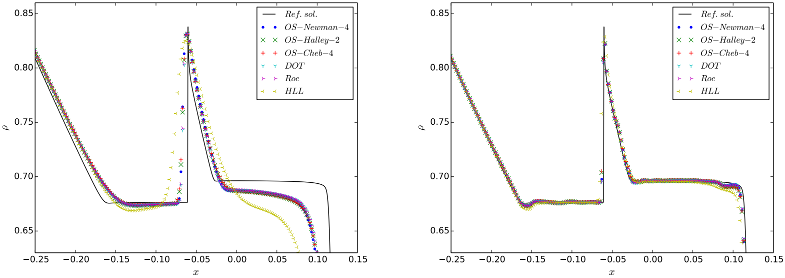


Figure: Zoom of the density compound wave. Left: first order. Right: third order.

Approximate OS solvers: Some examples

Brio-Wu shock tube problem

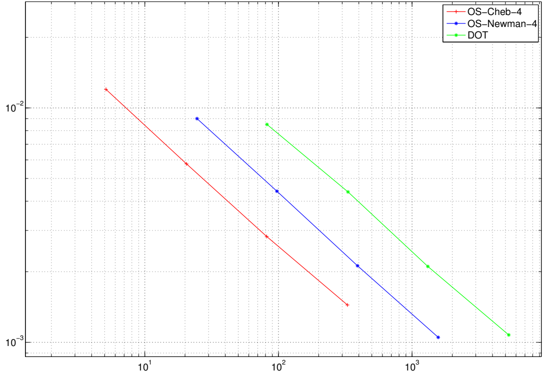


Figure: Efficiency curves (100, 200, 400, and 800 cells) (third order).

Approximate OS solvers: Some examples

Smooth isentropic vortex

- The purpose of this test, proposed by Hu and Shu, is to analyze the convergence and stability of the proposed numerical schemes.
- Initial condition: linear perturbation of an homogeneous state:

$$(\rho, v_x, v_y, P) = (1 + \delta\rho, 1 + \delta v_x, 1 + \delta v_y, 1 + \delta P).$$

- The exact solution is simply the initial condition convected by the mean velocity.
- Periodic boundary conditions and CFL=0.8.
- **Schemes:** High-order DOT, OS-Cheb-4, OS-Newman-4, and OS-Halley-2. All of them give similar results.

Approximate OS solvers: Some examples

Smooth isentropic vortex

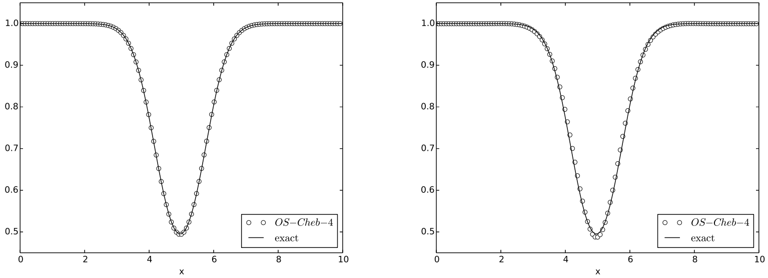


Figure: Density cut in the x-direction. Left: $t = 10$. Right: $t = 100$.

Approximate OS solvers: Some examples

Smooth isentropic vortex

	OS-Cheb-4		OS-Newman-4	
N	L^1 error	L^1 order	L^1 error	L^1 order
16	1.47E+00	–	1.47E+00	–
32	7.77E–01	0.92	7.95E–01	0.89
64	1.98E–01	1.97	2.03E–01	1.97
128	1.37E–02	3.85	1.39E–02	3.87

	OS-Halley-2		DOT	
N	L^1 error	L^1 order	L^1 error	L^1 order
16	1.46E+00	–	1.45E+00	–
32	7.81E–01	0.90	7.95E–01	0.87
64	1.95E–01	2.00	1.96E–01	2.02
128	1.33E–02	3.87	1.33E–02	3.88

Approximate OS solvers: Some examples

The Orszag-Tang vortex

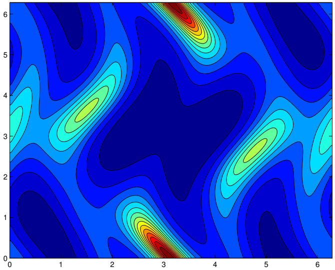
- Starting from a smooth state, the system develops complex interactions between different shock waves generated as the system evolves in the transition to turbulence.
- Initial data (Orszag and Tang, 1979): For $(x, y) \in [0, 2\pi] \times [0, 2\pi]$,

$$\begin{array}{lll} \rho(x,y,0) = \gamma^2, & v_x(x,y,0) = -\sin(y), & v_y(x,y,0) = \sin(x), \\ B_x(x,y,0) = -\sin(y), & B_y(x,y,0) = \sin(2x), & P(x,y,0) = \gamma, \end{array}$$

  with $\gamma = 5/3$.
- Periodic boundary conditions on a 192×192 uniform mesh and CFL=0.8.
- **Schemes:** DOT, OS-Cheb-4, OS-Newman-4, and OS-Halley-2.
- **High-order methods**: third-order compact WENO in space and third-order TVD Runge-Kutta in time.

Approximate OS solvers: Some examples

The Orszag-Tang vortex



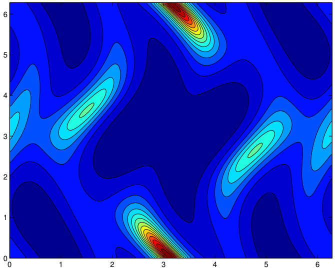


Figure: Density (left) and pressure (right) at time $t = 0.5$. Third-order OS-Cheb-4.

Approximate OS solvers: Some examples

The Orszag-Tang vortex

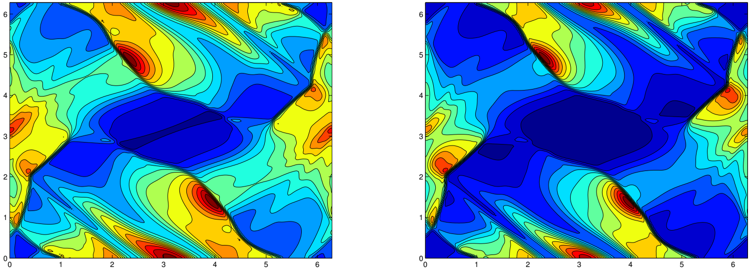


Figure: Density (left) and pressure (right) at time $t = 2$. Third-order OS-Cheb-4.

Approximate OS solvers: Some examples

The Orszag-Tang vortex

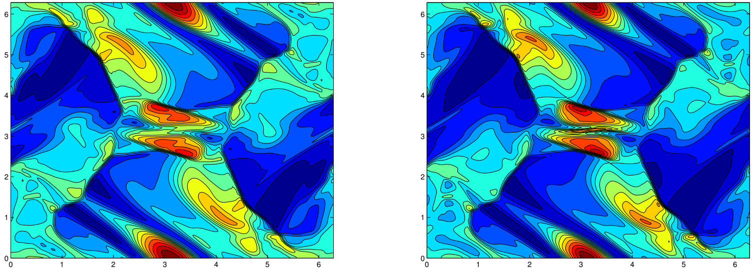


Figure: Density (left) and pressure (right) at time $t = 3$. Third-order OS-Cheb-4.

Approximate OS solvers: Some examples

The rotor problem

- Initially, there is a dense rotating disk at the center of the domain, while the ambient fluid remains at rest. These two areas are connected by means of a taper function, which helps to reduce the initial transient. Since the centrifugal forces are not balanced, the rotor is not in equilibrium. The rotating dense fluid will be confined into an oblate shape, due to the action of the magnetic field. (Balsara and Spicer, 1999).
- Periodic boundary conditions on a 200×200 uniform mesh and CFL=0.8.
- **Schemes:** DOT, OS-Cheb-4, OS-Newman-4, and OS-Halley-2.
- OS-Cheb-4, OS-Newman-4 and OS-Halley-2 give very similar results.

Approximate OS solvers: Some examples

The rotor problem

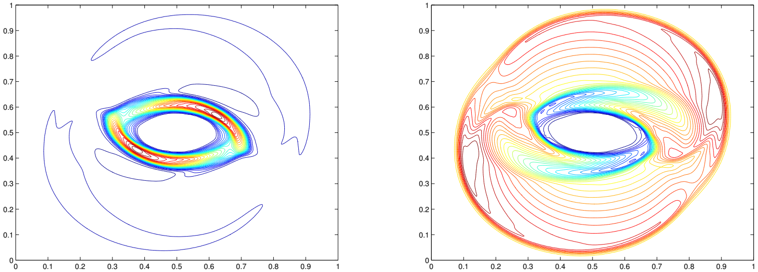


Figure: Left: density ρ. Right: pressure P. Third-order OS-Cheb-4 scheme at time $t = 0.295$.

Jacobian-free PVM-Chebyshev: Numerical tests

- Relativistic hydrodynamics (RHD) equations:

$$\boxed{\partial_t U + \partial_x F(U) = 0}$$

$$U = \begin{pmatrix} D \\ S \\ \tau \end{pmatrix}, \qquad F(U) = \begin{pmatrix} \dfrac{DS}{\tau + P + D} \\ \dfrac{S^2}{\tau + P + D} + P \\ \dfrac{(\tau + P)S}{\tau + P + D} \end{pmatrix}.$$

- Primitive variables (reference frame): ρ (density), u (velocity), ε (energy).
- Conserved variables (laboratory frame): D (mass density), S (momentum), τ (energy).
- Recovery:

$$D = \rho W, \quad S = \rho h W^2 u, \quad \tau = \rho h W^2 - P - \rho W.$$

$P = (\gamma - 1)\rho\varepsilon$: pressure (ideal gas); $W = \frac{1}{\sqrt{1-u^2}}$: Lorenz factor;
h: enthalpy; speed of light: $c = 1$.

Jacobian-free OS-Cheb: Numerical tests

Relativistic blast wave tests

- At $t = 0$ two regions of an ideal gas at rest are separated by a diaphragm which is suddenly removed.

- **Test 1**:

$$(\rho, u, P) = \begin{cases} (10, 0, 13.3) & \text{for } x \leqslant 0.5, \\ (1, 0, 0) & \text{for } x > 0.5, \end{cases}$$

  with $\gamma = 5/3$; $\Delta x = 1/800$, CFL=0.9, $T = 0.4$.

- **Test 2** (Norman-Winkler, 1986):

$$(\rho, u, P) = \begin{cases} (1, 0, 10^3) & \text{for } x \leqslant 0.5, \\ (1, 0, 10^{-2}) & \text{for } x > 0.5, \end{cases}$$

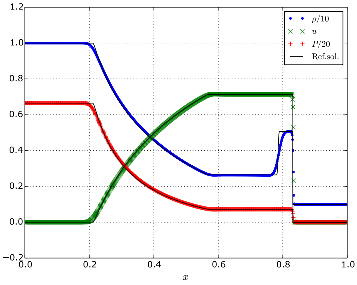
  with $\gamma = 5/3$; $\Delta x = 1/800$, CFL=0.9, $T = 0.35$.

- **Scheme:** Jacobian-free OS-Cheb-6. Third-order PHM in space and third-order TVD Runge-Kutta in time.

Jacobian-free OS-Cheb: Numerical tests

Relativistic blast wave tests

Test 1



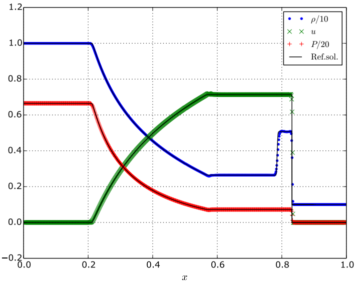
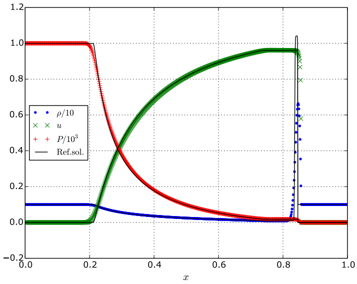


Figure: Normalized profiles of density, velocity and pressure. Left: first order; right: third order.

Jacobian-free OS-Cheb: Numerical tests

Relativistic blast wave tests

Test 2



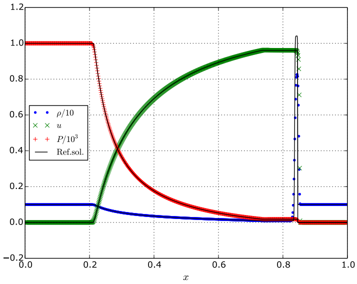


Figure: Normalized profiles of density, velocity and pressure. Left: first order; right: third order.

Lituya Bay event

- At 10:16pm (local time) on July 10, 1958. M_w 8.3 earthquake.
- Southwest sides and bottoms of Gilbert and Crillon inlets moved northwestward and relative to the northeast shore at he head of the bay, on the opposite side of the Fairweather fault.
- Shaking lasted about 4 minutes. Estimated total movements of 6.4m horizontally and 1 m vertically.
- About 2 minutes after the beginning of the earthquake the landslide was triggered.
- Slide volume estimated by Miller, 1960 in $30.6 \times 10^6\ m^3$.

Lituya Bay event

- Two-layer Savage-Hutter shallow-water model is used.
- A second order PVM-2U based on MUSCL reconstruction operator is used.
- A multi-GPU implementation is performed in order to speed-up the computation.

Picture from Weiss et. al., 2009

Lituya Bay 1958 event

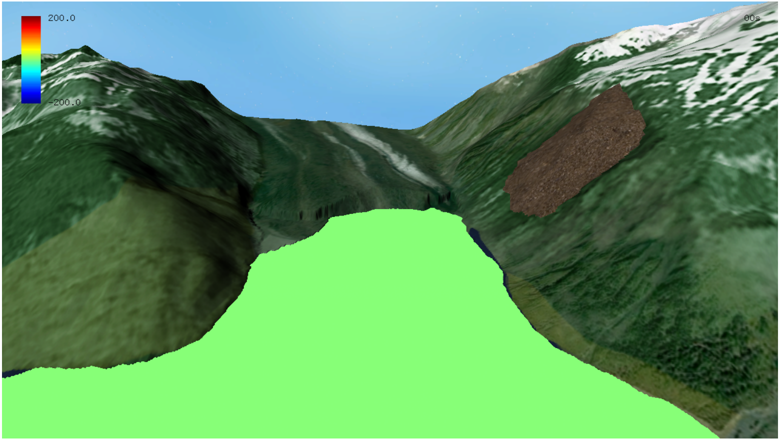


Figure: Lituya Bay event: Time $t = 0$ sec.

Lituya Bay 1958 event

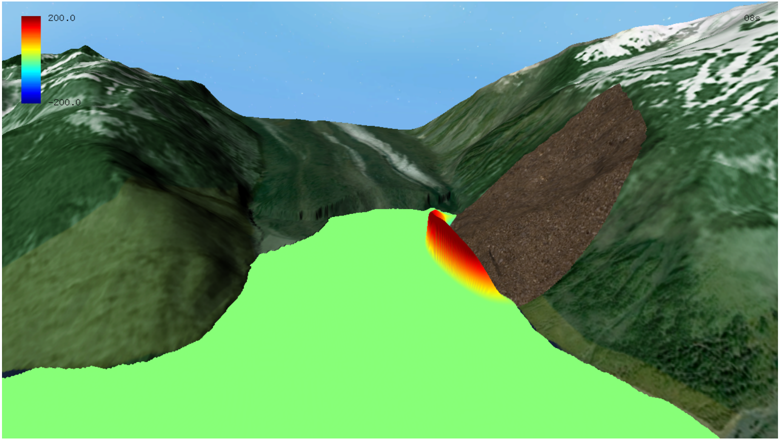


Figure: Lituya Bay event: Time $t = 8$ sec.

Lituya Bay 1958 event

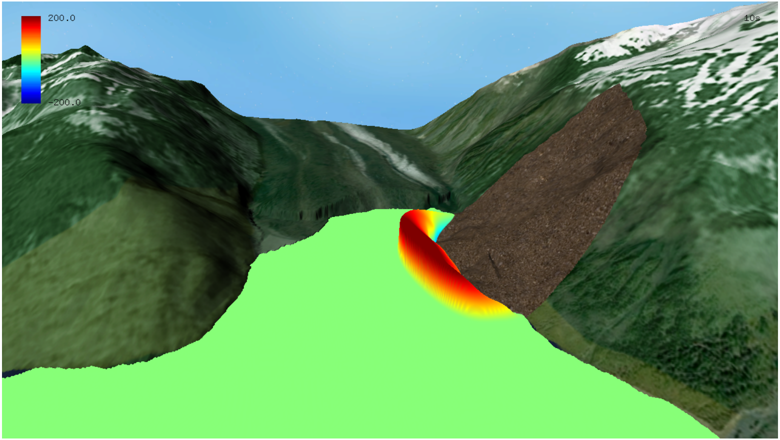


Figure: Lituya Bay event: Time $t = 10$ sec.

Lituya Bay 1958 event

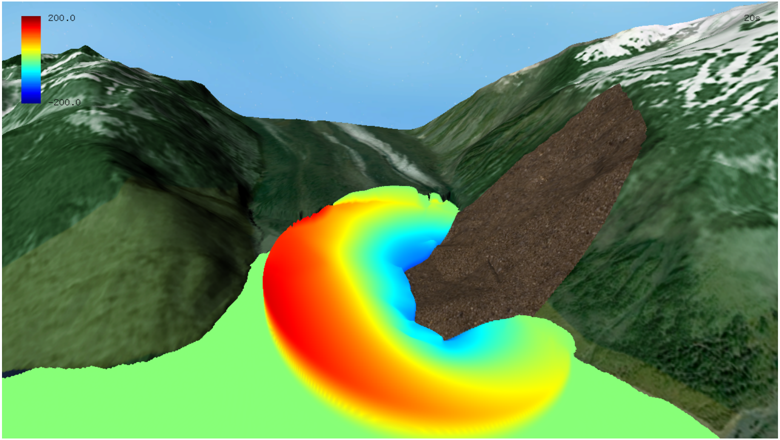


Figure: Lituya Bay event: Time $t = 20$ sec.

Lituya Bay 1958 event

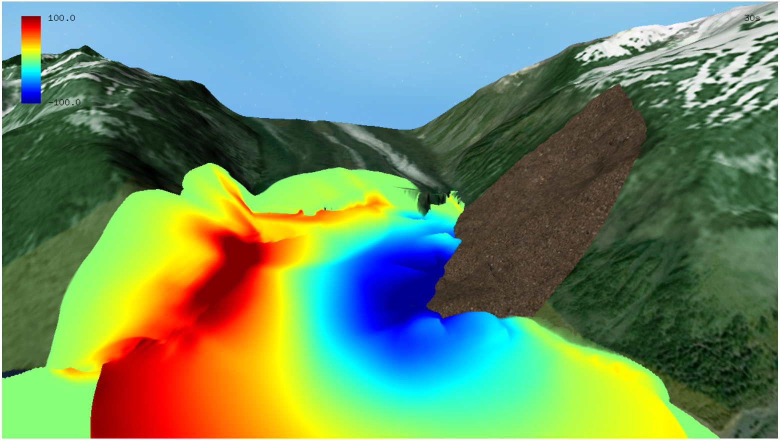


Figure: Lituya Bay event: Time $t = 30$ sec.

Lituya Bay 1958 event

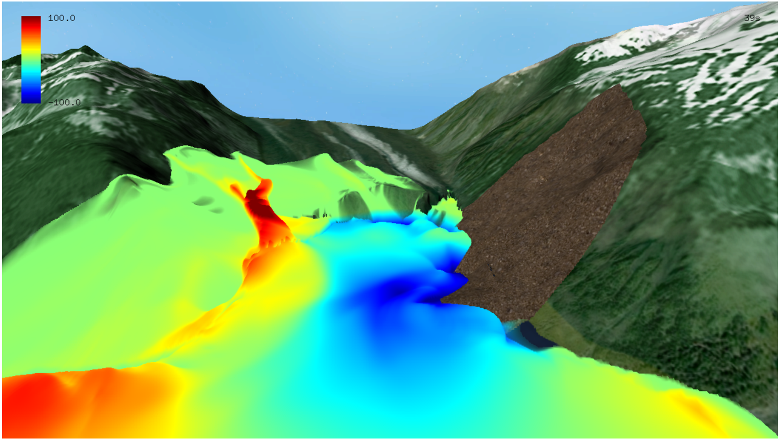


Figure: Lituya Bay event: Time $t = 39$ sec (max runup).

Lituya Bay event

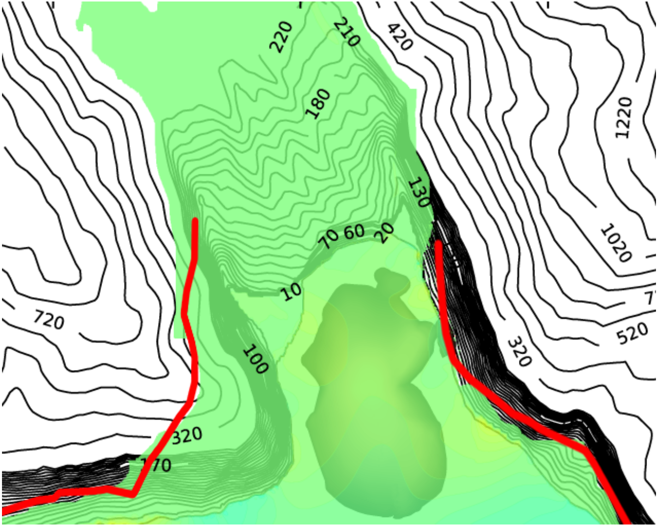


Figure: Lituya Bay event: estimated runup at Gilbert-Inlet

Lituya Bay event

Figure: Lituya Bay event: estimated runup at Cenotaph Island

Conclusions

- PVM, RVM and approximate OS methods provide an alternative to Roe's scheme when approximating time-dependent solutions in which the spectral decomposition is computationally expensive.
- Jacobian free implementation is also possible for some PVM solvers. This method is very easy to implement and could be used as and alternative method to other Jacobian free methods like Rusanov, Lax-Friedrich HLL, Force and GForce methods and, in general, intermediate waves can be precisely captured for an appropriate degree of approximation of the polynomial or rational function used.
- PVM, RVM and approximate OS solvers provide good results concerning precision and computational cost.
- PVM, RVM and approximate OS solvers could also be extended to balance laws and non-conservative problems using the path-conservative framework.

Thank you for your attention

Webpage:

http://edanya.uma.es

Youtube Channel:

http://youtube.com/grupoedanya