Solutions and Eigenvalues of Measure Differential Equations

Meirong Zhang

Department of Mathematical Sciences &
Zhou Pei-Yuan Center for Applied Mathematics,
Tsinghua University
(with G. Meng, Z. Wen, J. Qi, B. Xie et al)

Contents

I. Motivations

MDE is a special class of Generalized Differential Equations. MDE can be understood as Differential Equations with Measures as coefficients.

• Used in physics to model discontinuous, non-smooth, jump phenomena (or even the quantum effect).

• Mathematically, MDE is the limiting case of ODE/PDE.

Some problems unclear in ODE are much simpler in MDE.

General theory for Generalized ODE has been established, especially by the Prague school (Kurzweil, Schwabik et al.)

• Š. Schwabik, *Generalized Ordinary Differential Equations*, World Scientific, Singapore, 1992

• A. B. Mingarelli, *Volterra-Stieltjes Integral Equations and Generalized Ordinary Differential Expressions*, Lect. Notes Math., Vol. **989**, Springer, New York, 1983

A motivating example for MDE is as follows. In the history of sciences and mathematics, we have the first non-trivial differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \rho(x) y = 0, \qquad x \in I = [0, 1], \tag{1}$$

where $x,\ y$ are 1D, while $\rho(x)$ is non-constant.

- Spatial oscillation of 1D strings: $\rho(x)$ is the (non-negative) density.

This will lead to

- Weighted eigenvalue problems

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \tau\rho(x)y = 0, \qquad x \in I = [0,1]. \tag{2}$$

Here τ is the spectral parameter.

- Eigenvalue problems

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (\lambda + q(x))y = 0, \qquad x \in I. \tag{3}$$

Here λ is the spectral parameter and $q(x)$ is the potential.

With the Dirichlet boundary condition

$$(D): \qquad y(0) = y(1) = 0,$$

or, with the Neumann boundary condition

$$(N): \qquad y'(0) = y'(1) = 0,$$

the structures of eigenvalues of problems (2) and (3) are completely clear. For example, problem (2) admits a sequence of (positive) eigenvalues (or frequencies) $\tau_m^D = \tau_m^D(\rho)$, $m \in \mathbb{N}$, and a sequence of (non-negative) eigenvalues (or frequencies) $\tau_m^N = \tau_m^N(\rho)$, $m \in \mathbb{Z}^+ := \{0\} \cup \mathbb{N}$.

In the classical textbooks, one is concerned with continuous densities $\rho(x) \in C(I)$.

More generally, densities $\rho(x)$ are in the Lebesgue space $L^1(I)$. In this case, the distribution of mass

$$\mu_\rho(x) := \int_{[0,x]} \rho(s)\,\mathrm{d}s, \qquad x \in I,$$

is absolutely continuous (a.c.) on I.

Problems

1. When the distributions of masses become more and more singular like the completely singular (c.s.) distributions (e.g. Dirac distributions), how the oscillation of strings can be explained?

2. What is the eigenvalue theory for problems with general distributions?

These can be explained by Measure Differential Equations (MDE).

II. MDE: Solutions

Instead distributions, a more suitable mathematical notion is measures.

We recall the concept of (Radon) Measures. Let $I = [0,1]$ and

$$C(I) = \text{space of continuous real-valued functions on } I, \text{ with the supremum norm } \|\cdot\|_{C^0}.$$

The measure space on I is the dual space

$$\mathcal{M}_0(I) := (C(I), \|\cdot\|_{C^0})^*,$$

with the norm $\|\cdot\|_{\mathrm{var}}$ of total variation.

Riesz representation theorem $\mu \in \mathcal{M}_0(I)$ are those functions on I such that

- $\mu(x)$ is right-continuous on $(0,1)$,
- $\mu(x)$ has bounded variation on I: $\|\mu\|_{\mathrm{var}} < +\infty$,
- $\mu(x)$ is usually normalized as $\mu(0+) = 0$.

Examples of measures

1. For $q \in L^1(I)$,

$$\mu_q(x) := \int_0^x q(s)\,\mathrm{d}s, \qquad x \in I,$$

is an absolutely continuous (a.c.) measure on I w.r.t. the Lebesgue measure ℓ: $\ell(x) \equiv x$.

2. (Unit) Dirac measures δ_a, located at $a \in I$, are completely singular (c.s.). For $a = 0$,

$$\delta_0(x) = \begin{cases} -1 & \text{at } x = 0, \\ 0 & \text{for } x \in (0, 1]. \end{cases}$$

For $a \in (0,1]$,

$$\delta_a(x) = \begin{cases} 0 & \text{for } x \in [0,a), \\ 1 & \text{for } x \in [a,1]. \end{cases}$$

3. Singularly continuous (s.c.) measures: $\mu : I \to \mathbb{R}$ is continuous and

$$\mu'(x) = 0 \quad \ell\text{-a.e. } x \in I, \qquad \mu(I) \neq 0.$$

Arnold's Devil's Staircase: defined from dynamical systems. For parameters $\varepsilon \in [0, 1/2\pi]$ and $x \in I$, define a homeomorphism

$$\varphi_{\varepsilon,x} : \mathbb{R} \to \mathbb{R}, \qquad \theta \mapsto \theta + x + \varepsilon \sin(2\pi\theta).$$

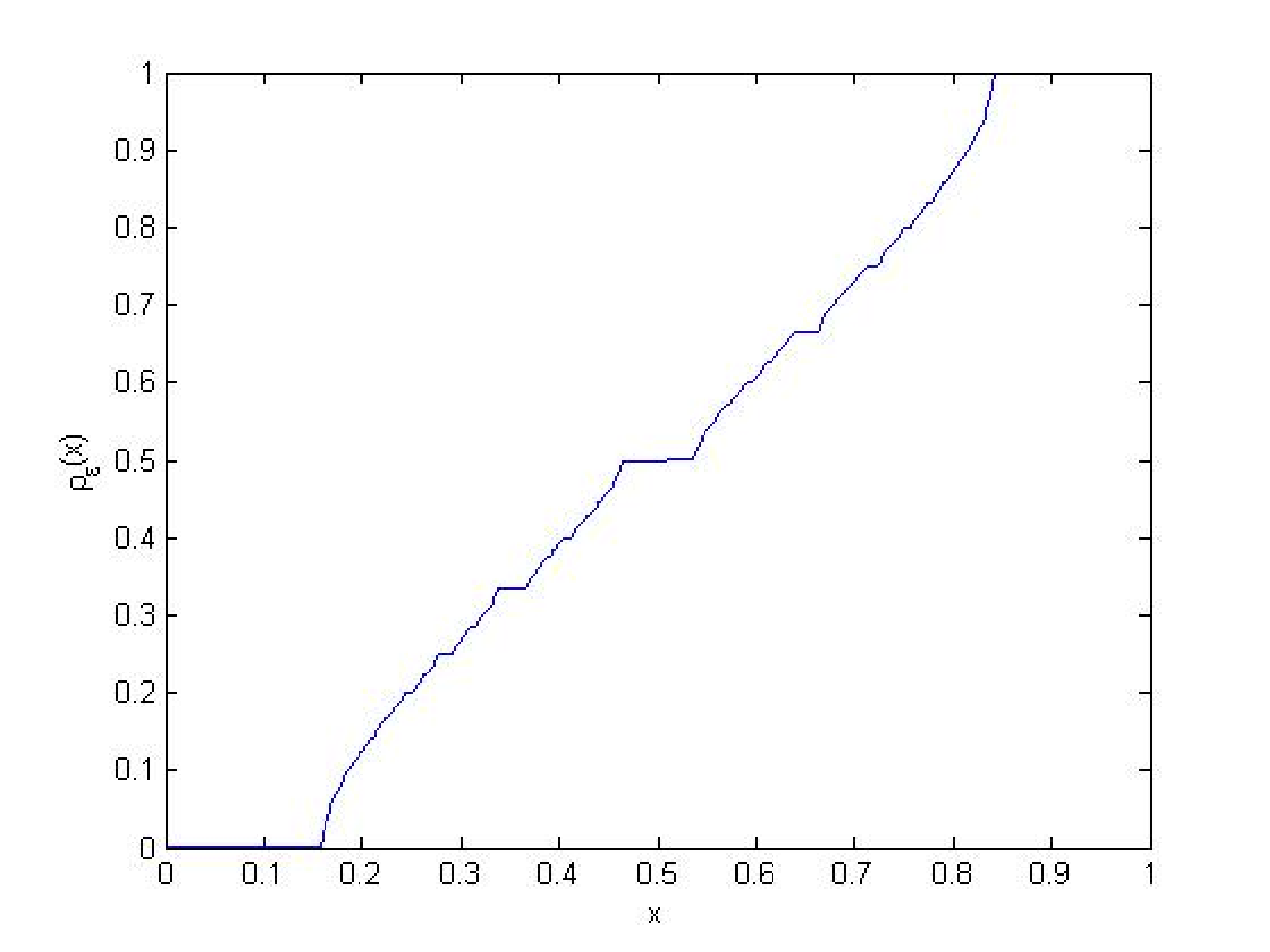
The rotation number of $\varphi_{\varepsilon,x}$ is

$$\varrho_\varepsilon(x) := \lim_{n \to +\infty} \frac{\varphi^n_{\varepsilon,x}(0)}{n} = \lim_{n \to +\infty} \frac{\varphi^n_{\varepsilon,x}(\theta) - \theta}{n} \quad \forall \theta \in \mathbb{R}.$$

(Independence of the initial values $\theta \in \mathbb{R}$)

As a function of $x \in I$,

- $\varrho_\varepsilon(x) \in C(I)$,
- $\varrho_\varepsilon(x)$ is non-decreasing on I,
- $\varrho_\varepsilon(0) = 0$ and $\varrho_\varepsilon(1) = 1$,
- $\varrho_0 = \ell$. In case $\varepsilon \in (0, 1/2\pi]$, $\varrho_\varepsilon^{-1}(r)$ is a non-trivial interval for each rational $r \in [0, 1]$
- $\varrho_\varepsilon(x)$ is an s.c. measure,
- by considering x as the standard time, $\varrho_\varepsilon(x)$ can be considered as a singular time.



Devil's staircase with $\varepsilon = 1/(2\pi)$.

Theorem 1. *(From real analysis) For 1D measure* $\mu \in \mathcal{M}_0(I)$*, one has the unique decomposition*

$$\mu = \mu_{ac} + \mu_{sc} + \mu_{cs}, \tag{4}$$

where $\mu_{sc}(x)$ *is s.c., and*

$$\mu_{ac}(x) = \int_{[0,x]} \rho(s)\,\mathrm{d}s, \qquad \mu_{cs}(x) = \sum_{a\in A} m_a \delta_a(x),$$

where $A \subset I$ *is at most countable and masses* $m_a \in \mathbb{R}$ *satisfy*

$$\sum_{a\in A} |m_a| < +\infty.$$

Integration

For $y \in C(I)$ and $\mu \in \mathcal{M}_0(I)$, the Riemann-Stieltjes integral

$$\int_I y \, \mathrm{d}\mu$$

is defined. For subintervals $J \subset I$, the Lebesgue-Stieltjes integral

$$\int_J y \, \mathrm{d}\mu$$

is also well defined.

2nd-order linear MDE

With a measure $\mu \in \mathcal{M}_0(I)$, the 2nd-order linear MDE is written in [15] (Meng & Zhang, JDE, 2013) as

$$\mathrm{d}y^{\bullet} + y\,\mathrm{d}\mu(x) = 0, \qquad x \in I. \tag{5}$$

The initial value (at $x = 0$) is

$$(y(0), y^{\bullet}(0)) = (y_0, v_0) \in \mathbb{R}^2 \quad (\mathbb{C}^2).$$

Formally, MDE (5) is equivalent to

$$\mathrm{d}y(x) = z(x)\,\mathrm{d}x, \qquad \mathrm{d}z(x) = -y(x)\,\mathrm{d}\mu(x).$$

The solution $y(x)$ and its generalized velocity $y^\bullet(x)$ of the IVP of (5) are determined by the system of integral equations

$$y(x) = y_0 + \int_{[0,x]} y^\bullet(s)\,\mathrm{d}s \quad \text{for } x \in I, \tag{6}$$

$$y^\bullet(x) = \begin{cases} v_0 & \text{for } x = 0, \\ v_0 - \int_{[0,x]} y(s)\,\mathrm{d}\mu(s) & \text{for } x \in (0,1]. \end{cases} \tag{7}$$

Remark

- If $\mu(x)$ is C^1, Eq. (7) is reduced to Riemann integral.
- If $\mu(x)$ is a.c., Eq. (7) is reduced to Lebesgue integral.
- For general measure μ, Eq. (7) is concerned with the Riemann-Stieltjes integral, while Eq. (6) is concerned with the Lebesgue integral.

Known results for linear MDE

- The IVP has the unique solution $(y(x), y^\bullet(x))$ on I.

- Solutions $y(x)$ are absolutely continuous in $x \in I$.

- Generalized velocities $y^\bullet(x)$ are non-normalized measures or BV-functions on I.

- At $x \in (0,1)$, $y^\bullet(x)$ coincides with the classical right-derivative of $y(x)$

$$y^\bullet(x) = \lim_{s \downarrow x} \frac{y(s) - y(x)}{s - x}.$$

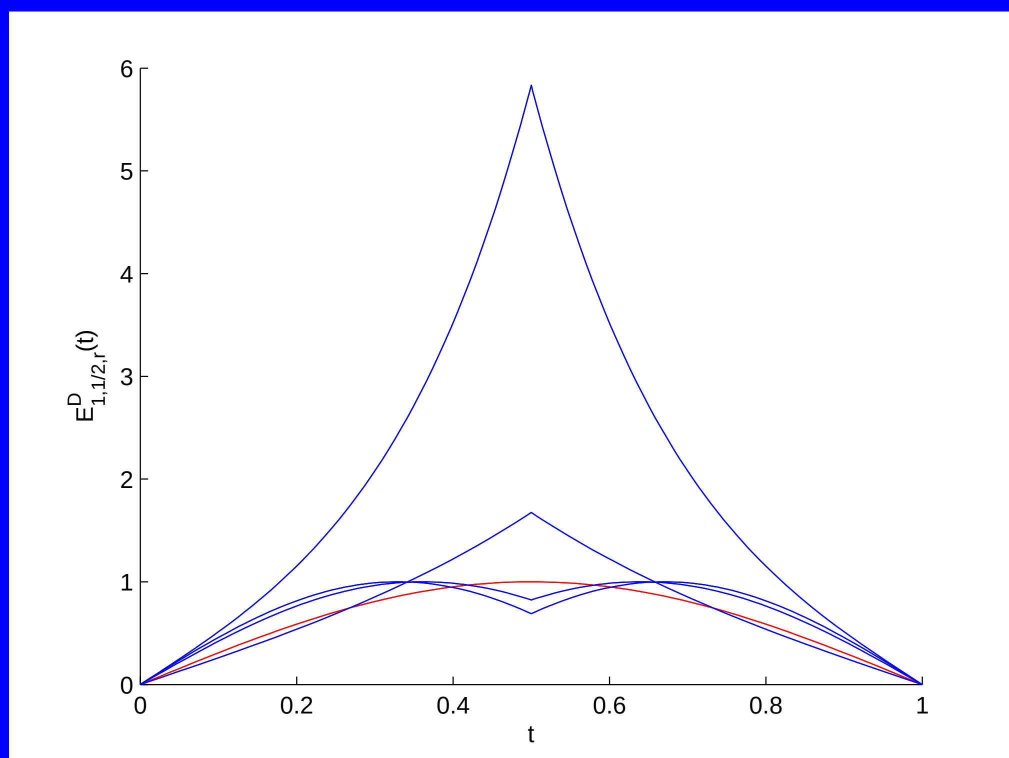
• In case $\Delta\mu(x_0) := \mu(x_0) - \mu(x_0-) \neq 0$, velocity $y^\bullet(x)$ has a jump or impulse at $x = x_0$

$$y^\bullet(x_0) - y^\bullet(x_0-) = -y(x_0) \cdot \Delta\mu(x_0).$$

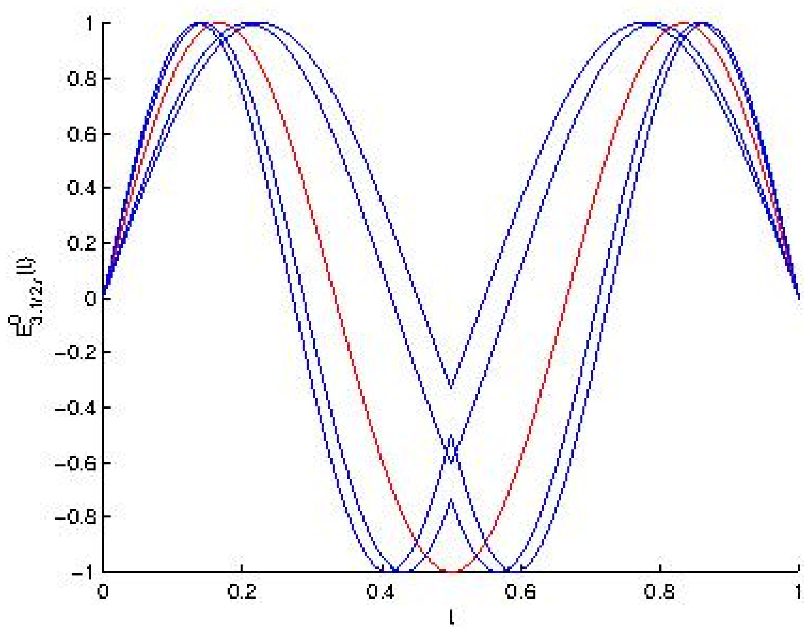
• MDE (5) is conservative: One has the Liouville law.

All proofs are obtained by integration, not by differentiation!

Some of the Dirichlet eigen-functions for $\mu = r\delta_{1/2}$ as potentials are as in the following figures: $r = 0$, $r > 0$ and $r < 0$.



6
5
4
3
2
1
0
0
0.2
0.4
0.6
0.8
1
t
$E^{D}_{1,1/2,r}(t)$



E^o_{3,1/2,r}(t)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.2
0.4
0.6
0.8
1
t

Comparisons with other types of differential equations

Stochastic Differential Equation (SDE):

The decomposition (4) for measures can be written as

$$\mu = \mu_{ac} + \mu_s, \qquad \text{where } \mu_s := \mu_{sc} + \mu_{cs}.$$

Then MDE (5) is

$$\mathrm{d}y^{\bullet} + \rho(x)y\,\mathrm{d}x + y\,\mathrm{d}\mu_s(x) = 0.$$

This similar to SDE, but with many types of singular measures $\mu_s(x)$.

Impulsive Differential Equation (IDE):

In (4), $\mu_{sc} = 0$ and $A \subset I$ is discrete. MDE (5) is

$$\mathrm{d}y^{\bullet} + \rho(x)y\,\mathrm{d}x + y\,\mathrm{d}\left(\sum_{a\in A} m_a \delta_a(x)\right) = 0.$$

IDE with impulses at all $a \in A$

$$y^{\bullet}(a) - y^{\bullet}(a-) = -m_a y(a).$$

Hence MDE (5) allows infinitely many impulses for velocity $y^{\bullet}(x)$, e.g., at all $x \in I \cap \mathbb{Q}$.

Difference Equation (DE):

In (4), $\mu_c := \mu_{ac} + \mu_{sc} = 0$. MDE (5) is

$$\mathrm{d}y^{\bullet} + y\,\mathrm{d}\left(\sum_{a\in A} m_a \delta_a(x)\right) = 0.$$

Solutions are piecewise linear. It is a Difference Equation or a system of algebraic equations.

Integral Equation (IE): In case $\mu = \mu_{sc}$ is singularly continuous,

$$\mathrm{d}y^{\bullet} + y\,\mathrm{d}\mu_{sc}(x) = 0$$

is an integral equation, which is not studied in ODE.

Differential Equation on Time Scale $\mathbb{T}$ $(\subset I)$ (DETS):

In (4), let $\mu_{sc} = 0$. On any gap interval (α, β) of $I \setminus \mathbb{T}$, set $\rho(x) = 0$. Then MDE (5) is

$$\mathrm{d}y^{\bullet} + \rho(x)y\,\mathrm{d}x + y\,\mathrm{d}\left(\sum_{a\in A} m_a\delta_a(x)\right) = 0.$$

This models some type of DETS.

Our results on solutions of MDE

• The usual topology on measures is induced by the norm $\|\cdot\|_{\mathrm{var}}$ of total variation: $(\mathcal{M}_0(I), \|\cdot\|_{\mathrm{var}})$ is a Banach space.

• The weak* topology w^* is defined as $\mu_k \to \mu$ iff

$$\int_I y \,\mathrm{d}\mu_k \to \int_I y \,\mathrm{d}\mu \qquad \forall y \in C(I).$$

Theorem 2. ([15]: Dependence of solutions of IVP on measures)

- *Continuous dependence: For solutions themselves,*

$$(\mathcal{M}_0(I), w^*) \ni \mu \to y(\cdot;\mu) \in (C(I), \|\cdot\|_{C^0})$$

is continuous;

- *for velocities,*

$$(\mathcal{M}_0(I), w^*) \ni \mu \to y^\bullet(\cdot;\mu) \in (\mathcal{M}(I), w^*)$$

is continuous; and

- *for ending velocities (at $x = 1$),*

$$(\mathcal{M}_0(I), w^*) \ni \mu \to y^\bullet(1;\mu) \in \mathbb{R}$$

is continuous.

- *Continuous Fréchet differentiability: At any time $x_0 \in I$,*

$$(\mathcal{M}_0(I), \|\cdot\|_{\mathrm{var}}) \ni \mu \to$$

$$(y(x_0;\mu), y^\bullet(x_0;\mu)) \in \mathbb{R}^2$$

is continuously differentiable.

The second result for velocities is optimal: For any $x_0 \in (0,1)$,

$$(\mathcal{M}_0(I), w^*) \ni \mu \to y^\bullet(x_0;\mu) \in \mathbb{R}$$

may NOT be continuous at some measure μ.

Most important ideas of the proof:

1. Transfer solutions of IVP to the fixed point of integral operator

$$y(x) = y_0 + v_0 x - \int_I G(x,s) y(s)\,\mathrm{d}\mu(s), \qquad x \in I,$$

where the kernel $G : I^2 \to \mathbb{R}$ is continuous:

$$G(x,s) = \begin{cases} x - s & \text{for } 0 \le s \le x \le 1, \\ 0 & \text{for } 0 \le x \le s \le 1. \end{cases}$$

2. Compactness argument from weak* topology.

3. For the optimal result on velocities, the reason is that

$$\mu_n \xrightarrow{w^*} \mu \text{ on } I$$

$$\not\Longrightarrow \mu_n|_J \xrightarrow{w^*} \mu|_J \text{ on } J,$$

where J is a subinterval of I. (This is different from the weak convergence in L^p spaces!) $\square$

III. MDE: Eigenvalue Theories

III1. Potentials are Measures

For an arbitrary measure $\mu(x)$, considered as a potential, one has the corresponding Eigenvalue Problem

$$\mathrm{d}y^{\bullet} + y\,\mathrm{d}\mu(x) + \lambda y\,\mathrm{d}x = 0, \qquad x \in I. \tag{8}$$

With the Dirichlet or the Neumann boundary conditions

$$(D): y(0) = y(1) = 0, \qquad (N): y^{\bullet}(0) = y^{\bullet}(1) = 0,$$

the basic eigenvalue theory has been obtained in [15].

As for the Structure of eigenvalues,

Theorem 3. (Meng & Zhang, JDE, 2013) *Structures of MDE (7) are the same as the classical Sturm-Liouville problems of ODE:*

$$\left\{\lambda_i^D(\mu)\right\}_{i\in\mathbb{N}}, \quad \left\{\lambda_i^N(\mu)\right\}_{i\in\mathbb{Z}^+}, \quad \lambda_i(\mu)\to+\infty.$$

As for the Dependence of eigenvalues on measures,

Theorem 4. (Meng & Zhang, JDE, 2013)

• *$\lambda_i^{D/N}(\mu)$ of MDE are continuously Fréchet differentiable in measures $\mu \in (\mathcal{M}_0(I), \|\cdot\|_{\rm var})$. Moreover,*

$$\partial_\mu \lambda_i^{D/N}(\mu) = -\left| E_i^{D/N}(\cdot;\mu) \right|^2,$$

where $E_i^{D/N}(x;\mu)$ are normalized eigen-functions associated with $\lambda_i^{D/N}(\mu)$.

• *$\lambda_i^{D/N}(\mu)$ of MDE are continuous in measures $\mu \in (\mathcal{M}_0(I), w^*)$.*

- The strongest continuous dependence!

- For the classical Sturm-Liouville problems, one has the strong continuity of eigenvalues in integrable potentials/weights. See, for example, J. Pöschel and E. Trubowitz (*The Inverse Spectral Theory*, Academic Press, New York, 1987) for a preliminary result, and our works [1, 2, 3, 4, 6] for general cases of 2nd or 4th-order problems.

Main ideas of the proof

A novel definition for the Prüfer transformation and the arguments of MDE.

1. Recall that, for eigenvalue problem

$$y'' + (\lambda + q(x))y = 0, \qquad x \in [0, 1],$$

the simplest approach is to introduce the argument $\theta(x)$ by the Prüfer transformation

$$y = r\sin\theta, \qquad y' = r\cos\theta.$$

For ODE case, $\theta(x)$ is (absolutely) continuous in $x \in [0,1]$ and is determined by nonlinear ODE

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \cos^2\theta + (\lambda + q(x))\sin^2\theta, \qquad x \in [0,1].$$

For MDE (8), it seems that the corresponding argument θ is defined using 1st-order nonlinear MDE

$$\mathrm{d}\theta = \cos^2\theta\,\mathrm{d}x + \sin^2\theta\,\mathrm{d}\left(\lambda x + \mu(x)\right), \qquad x \in [0,1].$$

This is hopeless, because both $\sin^2\theta(x)$ and $\lambda x + \mu(x)$ may be discontinuous.

In order to introduce the arguments for MDE (7), we need some topological idea, with the help of the continuous dependence of solutions on measures in Theorem 2.

2. Given $\mu \in \mathcal{M}_0[0,1]$ and λ, by introducing a homotopy parameter $\tau \in [0,1]$, we consider MDE

$$\mathrm{d}y^{\bullet}(x) + y(x)\,\mathrm{d}(\lambda x + \tau\mu(x)) = 0, \qquad x \in [0,1].$$

3. The solutions and velocities define linear transformations $M(x; \lambda\ell + \tau\mu)$ on $\mathbb{R}^2$, which can be reduced to transformations $\hat{M}(x; \lambda\ell + \tau\mu)$ on the unit circle $\mathbb{S}^1$.

4. Given $x \in [0, 1]$, $\hat{M}(x; \lambda\ell + \tau\mu)$ is continuous in $\tau \in [0, 1]$. The topological lifting of $\hat{M}(x; \lambda\ell + \tau\mu)$ to $\mathbb{R}$ is then well-defined once the covering mapping of $\hat{M}(x; \lambda\ell + 0 \cdot \mu)$ (a simple ODE) is chosen (as the standard one).

By taking $\tau = 1$, the argument of MDE (7) is defined as the mapping $\theta(x; \vartheta, \lambda\ell + \mu)$ on $\mathbb{R}$.

5. Eigenvalues λ of (8) with (D) or with (N) are then determined by equations

$$\theta(1; 0, \lambda\ell + \mu) = m\pi, \qquad m \in \mathbb{N},$$

or

$$\theta(1; \pi/2, \lambda\ell + \mu) = m\pi + \pi/2, \qquad m \in \mathbb{Z}^+ = \{0\} \cup \mathbb{N}.$$

6. To obtain the structure of eigenvalues, estimates for $\theta(1; \theta_0, \lambda\ell + \mu)$, as $\lambda \to \pm\infty$, can be done as for ODE. $\square$

III2. Weights are Measures

The weighted eigenvalue problem with the semi-positive weighted measure ω is

$$\mathrm{d}y^{\bullet} + \tau y \, \mathrm{d}\omega(x) = 0, \qquad x \in I. \tag{9}$$

Here $\omega \in \mathcal{M}_0(I)$ be a semi-positive measure, i.e.,

• $\omega(x) : I \to \mathbb{R}$ is non-decreasing, • $\omega(I) > 0$.

In this case, ω_{ac}, ω_{sc}, and ω_{cs} in the decomposition (4) are also non-decreasing.

In a preprint [17], we obtain some new results on eigenvalues. The first is

Theorem 5. *([17]) Suppose that* $\omega = \omega_{cs}$ *and* ω *contains precisely* $n \in \mathbb{N}$ *Dirac measures inside* $(0, 1)$. *Then, with* (D), *problem (9) admits precisely* n *weighted eigenvalues* $\tau_i^D(\omega)$, $1 \le i \le n$.

This can be reduced to a system of linear equations in $\mathbb{R}^n$.

The second is that the converse of Theorem 5 is also true.

Theorem 6. *([17]) With (D), problem (9) admits precisely n weighted eigenvalues $\tau_i^D(\omega)$, $1 \leq i \leq n$, iff ω contains precisely $n \in \mathbb{N}$ Dirac measures inside the interior $(0,1)$. In other words, the number of the Dirichlet weighted eigenvalues of (9) is*

$$K_{\omega,D} = \begin{cases} +\infty & \text{if } \omega_c = \omega_{ac} + \omega_{sc} \neq 0, \\ \#(A \cap (0,1)) & \text{if } \omega = \sum_{a \in A} m_a \delta_a, \end{cases}$$

where $m_a > 0$ for all $a \in A$.

Theorem 7. *([17]) With (N), the number of the Neumann weighted eigenvalues of (9) is*

$$K_{\omega,N} = \begin{cases} +\infty & \textit{if } \omega_c \neq 0, \\ \#A & \textit{if } \omega = \sum_{a\in A} m_a \delta_a, \end{cases}$$

where $m_a > 0$ for all $a \in A$.

For example, for

$$\omega = \sum_{n=1}^{\infty} \frac{1}{n^2} \delta_{1/n},$$

both the Dirichlet and the Neumann have ∞ many weighted eigenvalues.

Ideas of the proof

1. For non-zero s.c. measures, the structure of weighted eigenvalues of MDE is the same as that of ODE with definite integrable weights.

2. The crucial observation is that: for s.c. $\omega > 0$, the argument

$$\begin{cases} \mathrm{d}\Theta(x,\tau) = \cos^2\Theta(x,\tau)\,\mathrm{d}x + \tau\sin^2\Theta(x,\tau)\,\mathrm{d}\omega(x) \\ \Theta(0) = 0 \end{cases}$$

satisfies $\Theta(1,\tau) \to +\infty$ as $\tau \to +\infty$.

3. The proof is that all non-c.s. measures always admit infinitely many weighted eigenvalues. $\square$

The third is some sharp strong continuity of weighted eigenvalues of MDE in measures.

Theorem 8. *([17]) Suppose that semi-positive measures $\omega_k \to \omega$ in $(\mathcal{M}_0(I), w^*)$. One has*

- $\liminf_{k\to\infty} K_{\omega_k} \geq K_\omega$,
- *for any* $1 \leq i \leq K_\omega$, *there holds* $\lim_{k\to\infty} \tau_i(\omega_k) = \tau_i(\omega)$.
- *the normalized weighted eigen-functions* $E_i(\cdot;\omega)$ *are also strong continuous in* ω.

IV. Applications and Problems

1. Lyapunov stability criterion

For 1-periodic Hill's equations,

$$q(t) > 0, \int_0^1 q \leq 4 \Longrightarrow \ddot{y} + q(t)y = 0 \text{ is stable.}$$

Here we have the non-strict inequality $\leq$ and the optimal constant 4.

Using the weighted eigenvalues of MDE, there is some connection with the Dirac measure $\omega = \delta_{1/2}$. The unique weighted Dirichlet eigenvalue is

$$\tau_1 = 4,$$

with the normalized eigen-function

$$\begin{aligned} E_1(x) &= \sqrt{12} \cdot \min\{x, 1-x\}, \qquad x \in [0,1], \\ &= \sqrt{12} \cdot \operatorname{dist}(x, \mathbb{Z}), \qquad x \in \mathbb{R}. \end{aligned}$$

The optimal Lyapunov criterion cannot be realized by ODE, but by MDE.

2. Extremal eigenvalues

The Banach-Alaglou theorem implies that bounded subsets of $\mathcal{M}_0(I)$ (in $\|\cdot\|_{\rm var}$) are relatively sequentially compact (in the weak* topology w^*).

As a consequence of eigenvalues in measures,

$$\min\{\lambda_1^D(\mu) : \mu \in \mathcal{M}_0(I),\ \|\mu\|_{\rm var} \le r\} = \mathbf{L}_1(r)$$

can be realized by some measure. In fact, one has

$$\mathbf{L}_1(r) = \lambda_1^D(r\delta_{1/2}).$$

Note that

$$\inf\{\lambda_1^D(q) : q \in L^1(I),\ \|q\|_1 \leq r\} = \mathbf{L}_1(r),$$

which is not realized by any potential.

3. Approximation of eigenvalues: ODE vs MDE

Dirac measures are approximated by smooth measures (in the weak* topology). For example, given a c.s. measure

$$\omega = \omega_{cs} = \sum_{i=1}^{n} m_i \delta_{a_i},$$

one has some smooth functions (measures) $\omega_k \in C^\infty(I) \cap \mathcal{M}_0(I)$ such that $\omega_k \to \omega_{cs}$ in $(\mathcal{M}_0(I), w^*)$.

The continuity result in Theorem 8 means

$$\lim_{k\to\infty} \tau_i(\omega_k) \text{ (ODEs)} = \tau_i(\omega_{cs}) \text{ (DE)}, \quad 1 \le i \le n.$$

Conversely, given any measure $\omega \in \mathcal{M}_0(I)$, say a smooth measure, define, for $k \in \mathbb{N}$,

$$\omega_k := \sum_{j=1}^{k} m_{k,j}\delta_{j/k}, \qquad m_{k,j} := \omega(I_{k,j}),$$

where $I_{k,1} := [0, 1/k]$ and $I_{k,j} := ((j-1)/k, j/k]$ for $2 \leq j \leq k$. Then ω_k are c.s. measures. It is easy to verify that $\omega_k \to \omega$ in $(\mathcal{M}_0(I), w^*)$ as $k \to \infty$.

By the continuity of Theorem 8 again, for any $i \in \mathbb{N}$, there holds

$$\lim_{k\to\infty} \tau_i(\omega_k) \text{ (DEs)} = \tau_i(\omega) \text{ (ODE)}.$$

The left-hand side is algebraic problems, while the right-hand side is an ODE problem.

- For eigenvalues, algebraic problems and ODE problems can be mutually approximated.
- For s.c. measures like the Devil's staircases, what are the eigenvalues and the dynamics?
- New approach to inverse spectral problems?

4. Orthogonal systems deduced from MDE

Given $\mu \in \mathcal{M}_0(I)$, it can be proved that $\{E_i(\cdot;\mu)\}$ forms an orthogonal system for $L^2(I)$.

Problem: Is the orthogonal system $\{E_i(\cdot;\mu)\}$ complete in $L^2(I)$?

If yes, like the Fourier expansion, we may use them to effectively expand functions with jumps, like wavelets.

Thank you

Literatur

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